Probably Approximately Correct (PAC) Selection in Simulation/Best-Arm Problems

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Problem Setting

- Finite number of alternatives, i.e., arms.
- Optimize a scalar performance measure of interest.
- An alternative’s performance is observed with simulation noise.

Examples:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Performance Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>hospital bed allocation</td>
<td>expected diversion costs</td>
</tr>
<tr>
<td>ambulance base location</td>
<td>expected call response time</td>
</tr>
<tr>
<td>MDP policy</td>
<td>expected discounted total cost</td>
</tr>
</tbody>
</table>
Assumptions

Alternative 1 | $X_{11}$ | $X_{12}$ | ... | i.i.d., $\sim F_1$ with mean $\mu_1$
Alternative 2 | $X_{21}$ | $X_{22}$ | ... | i.i.d., $\sim F_2$ with mean $\mu_2$

Assume $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$, where the order is unknown.

Observations across alternatives are independent.

- Unless CRN used for variance reduction.

Marginal distributions $F_i$:

- R&S: Normal distribution
- MAB: Bounded support or sub-Gaussian distribution
Selection Procedures

Typical Procedure

1. Obtain observations to estimate alternatives’ performances.
   - Calculate estimators $Y_1, \ldots, Y_k$ of $\mu_1, \ldots, \mu_k$.
2. Select the alternative with the best estimated performance.
   - Select alternative $K := \text{arg max } Y_i$.

Would like to take as few samples as possible.

Most efficient procedures use *screening* to eliminate inferior systems.
Objective

PAC Selection Guarantee

A type of *fixed-confidence* guarantee on the performance of the chosen alternative *relative* to the other alternatives.

Probably Approximately Correct (PAC) election

\[ \text{w.p. } 1 - \alpha \text{ within } \delta \text{ of the best} \]

“Close enough is good enough.”

- Frequentist ranking and selection (R&S) \( \rightarrow \) known as PGS.
- Multi-armed bandits (MAB) in full exploration.
Proving PAC Selection Guarantees

MAB
- Concentration inequalities, e.g., Hoeffding, Chernoff.

R&S
- Multiple comparisons with the best (MCB).
- Often hard to prove directly for sequential procedures.
  - Session MB57 – “An Efficient Fully Sequential Procedure Guaranteeing Probably Approximately Correct Selection”
- A more common guarantee deals with correct selection.
Bechhofer (1954) developed the idea of an indifferece zone (IZ).

IZ parameter $\delta > 0$ is often described as the smallest difference in performance worth detecting.

- **Preference Zone**: $\text{PZ}(\delta) = \{ \mu : \mu_k - \mu_{k-1} \geq \delta \}$
  “The best alternative is at least $\delta$ better than all the others.”

- **Indifference Zone**: $\text{IZ}(\delta) = \{ \mu : \mu_k - \mu_{k-1} < \delta \}$
  “There are close competitors to the best alternative.”
Space of Configurations

E.g., for \( F_i := \mathcal{N}(\mu_i, \sigma_i^2) \):
Two Frequentist Guarantees

Let $K$ be the index of the chosen alternative. For specified confidence level $1 - \alpha \in (1/k, 1)$ and $\delta > 0$, guarantee

$$
\mathbb{P}_\mu(\mu_K > \mu_k - \delta) \geq 1 - \alpha \quad \text{for all } \mu, \quad \text{(Goal PACS)}
$$

$$
\mathbb{P}_\mu(\mu_K = \mu_k) \geq 1 - \alpha \quad \text{for all } \mu \in PZ(\delta). \quad \text{(Goal PCS-PZ)}
$$

Goal PACS $\implies$ Goal PCS-PZ.

Goal PCS-PZ is the standard in the frequentist R&S community, but doesn’t appear in the MAB literature.
Goal PCS-PZ vs Goal PACS

“Goal PCS-PZ is weaker, but is that so bad?”

Issues with Goal PCS-PZ

- Says nothing about performance in IZ(δ).
- Configurations in PZ(δ) may be unlikely in practice.
  - Large number of alternatives.
  - Alternatives found from search.
- Choice of δ restricts the problem.
- May require Bayesian belief about μ.

Goal PACS has none of these issues!
Equivalence of Goals

When does Goal PCS-PZ $\Rightarrow$ Goal PACS?

Intuition: More good alternatives, more likely to pick a good alternative.

Scattered results dating back to Fabian (1962), though none in the past 20 years.

Reasons for studying this:

- Show that R&S procedures meet Goal PACS.
- Determine how MAB procedures might be designed for Goal PCS-PZ, as a means to achieve Goal PACS.
Main Equivalence Results: Condition 1

Condition 1 (Guiard 1996)

For all subsets \( A \subset \{1, \ldots, k\} \), the joint distribution of the estimators \( Y_i \) for \( i \in A \) does not depend on \( \mu_j \) for \( j \notin A \).

“Changing the mean of an alternative doesn’t change the distribution of other alternatives’ estimators.”

Limitation: Can only be applied to procedures without screening.

- Normal (i.i.d.): Bechhofer (1954), Dudewicz and Dalal (1975), Rinott (1978)
- Bernoulli: Sobel and Huyett (1957)
- Support \([a, b]\): Naive Algorithm of Even-Dar et al. (2006)
Main Equivalence Results: Condition 2

Condition 2 (Hayter 1994)
For all alternatives $i = 1, \ldots, k$,

$$\mathbb{P}_\mu(\text{Select alternative } i)$$

is non-increasing in $\mu_j$ for every $j \neq i$.

“Improving an alternative doesn’t help any other alternative get selected.”

Limitation: Deriving an expression for $\mathbb{P}_\mu(\text{Select alternative } i)$ is hard.
Main Equivalence Results: Condition 2

Procedure not satisfying Condition 2

1. Take \( n_0 \) samples of each alternative.
2. Eliminate all but the two alternatives with the highest means.
3. Take \( n_1 \) additional samples for the two surviving alternatives.
4. Select the surviving alternative with the highest overall mean.

Consider the three-alternative case: \( \mu_1 < \mu_2 < \mu_3 \).
- Track \( P_\mu(\text{Select alternative 2}) \) as \( \mu_1 \) increases up to \( \mu_2 \).
- Consider \( n_1 = 0 \) and \( n_1 = \infty \) as extreme cases.
Main Equivalence Results: Condition 3

Condition 3
For all alternatives $i = 1, \ldots, k$,

$$\mathbb{P}_\mu(\text{Select alternative } j, \text{ for some } j < i)$$

is non-increasing in $\mu_i$.

“Improving an alternative doesn’t help inferior alternatives get selected.”

Condition 2 $\Rightarrow$ Condition 3.
Conclusions

Main take-aways

- Goal PACS is superior to Goal PCS-PZ.
- Goal PACS can follow immediately from Goal PCS-PZ.
- Condition 3 has the potential to hold for many procedures, if only it could be verified.

Do modern sequential selection procedures achieve Goal PACS?

- KN of Kim and Nelson (2001)
- BIZ of Frazier (2014)

Can MAB procedures be designed for Goal PCS-PZ while also satisfying one of these conditions?
Questions
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