Optimal Pinging Frequencies in the Search for an Immobile Beacon

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Abstract:

We consider a search for an immobile object that can only be detected if the searcher is within a given range of the object during one of a finite number of instantaneous detection opportunities, i.e., "pings." More specifically, motivated by naval searches for battery-powered flight data recorders of missing aircraft, we consider the trade-off between the frequency of pings for an underwater locator beacon and the duration of the search. First, assuming the search speed is known, we formulate a mathematical model to determine the pinging period that maximizes the probability that the searcher detects the beacon before it stops pinging. Next, we consider generalizations to discrete search speed distributions under a uniform beacon location distribution. Lastly, we present a case study based on the search for Malaysia Airlines Flight 370 that suggests the industry-standard beacon pinging period—roughly one second between pings—is too short.

Keywords: immobile hider, linear search problem, definite range law, periodic detection

1 Introduction

Consider an optimal search problem in which an immobile object is hidden and is only detectable during a finite number of instantaneous opportunities, i.e., "pings." At each ping, the object is detected if the searcher is within a given radius of the object's location; otherwise it remains undetected. We consider the pinging frequency that maximizes the probability of finding the object before the last ping, which involves a trade-off between making detection opportunities more frequent and extending the duration of the search.

A motivating application is deep-sea searches for missing aircraft. All modern commercial aircraft carry "black box" flight data recorders (FDRs) that record the aircraft's performance parameters during flight (Federal Aviation Administration, 2007). Locating the FDR among aircraft wreckage is critical, as its recordings can help investigators determine the cause of the accident. To facilitate locating the device, each FDR is equipped with an underwater locator beacon (ULB) that activates when submerged in water (Kelland, 2009). Once activated, the ULB produces a series of ultrasonic pings, roughly once per second, and is required to have sufficient battery life to sustain 30 days of pinging. Once the beacon's battery fails, searchers must rely on less-effective methods such as high-altitude fly-overs and slower methods such as side-scan sonar searches conducted by minisubmarines (Stone et al., 2011; Russia Today, 2014). In the case of Air France Flight 447, failure to find the FDR while its ULB was pinging led to a two-year, $40 million search effort before the device was finally found (Russia Today, 2014).

In practice, while the beacon is actively pinging the search is carried out by multiple search vessels carrying towed pinger locators (TPLs). These sophisticated devices consist of a long cable connected to a towfish that houses a hydrophone, a passive listening device that can detect the beacon's pings. Each search vessel tows the towfish approximately 1,000 feet above the ocean floor while passing back and forth on parallel runs (Ahlers, 2014). Because the towfish is towed at such depths, each search vessel travels at only approximately 2.3 miles per hour and takes between 3–8 hours to turn around between runs. The search continues in this fashion until the FDR's position has been "boxed in," after which another set of perpendicular runs is performed over the reduced search area. Once the searchers are able to triangulate the beacon's location from the strength and location of detected pings, the FDR is recovered either by divers, submersibles, or remotely operated vehicles, depending on the depth and sea floor terrain (Kelland, 2009).

Beacon design is a timely and controversial issue within the aviation community as recent searches for missing aircraft have underscored the urgency of locating the FDR. Following the investigation of Air France Flight 447 in 2009, the Bureau d'Enquêtes et d'Analyses pour la Sécurité de l'Aviation Civile (BEA) proposed design modifications for ULBs to facilitate the search for FDRs (Bureau d'Enquêtes et d'Analyses pour la sécurité de l'aviation civile, 2009). A leading suggestion of the BEA was to add a second beacon that transmits at a lower frequency of sound, providing an increased range of detection of up to eight miles, compared to the current pinger range of 2.5 miles (Associated Press, 2014). Another recommendation, now strongly advocated by international bodies such as the European Aviation Safety Agency and International Civil Aviation Organization following the initial search efforts for Malaysia Airlines Flight 370, is to extend the beacon's battery life from 30 to 90 days (Hepher, 2014; Lowy , 2014). The model described herein could easily incorporate either of these modifications. While these design enhancements may increase the probability of a successful search, they are also costly compared to the alternative we consider: modifying the beacon's pinging period, i.e., the time between successive pings.

Optimal search theory has been applied in many searches for missing ships, submarines, and planes (Stone, 1992; Richardson and Stone, 1971; Stone et al., 2011) and dates back to the efforts of the Anti-Submarine Warfare Operations Research Group during World War II (Koopman, 1956a,b, 1957). Dobbie (1968), Stone (1975), and Benkoski et al. (1981) provide thorough surveys of early search theory results while Washburn (2002) is a comprehensive reference on search models and searches for a moving target. More recent work in search theory focuses on applications of game theory (Alpern and Lidbetter, 2013, 2014) and Markov decision processes (Kagan and Ben-Gal, 2013). Alpern and Gal (2003) models searcher games with a stationary object located in a network, multidimensional region, or collection of discrete locations. Alpern and Lidbetter (2015) consider a search game on a network in which the searcher chooses between moving at a fast or slow speed, but is more likely to observe the hider when searching at the slow speed.

In the well-studied one-dimensional linear search problem (LSP) introduced in Bellman (1963) and Beck (1964), a searcher searches for an object that is always detectable and located on the real line according to a known probability distribution. The searcher's objective is to determine the optimal sequence of turning points—referred to as a search plan—that minimizes the expected time until the searcher reaches the object's location. Franck (1965) provides necessary and sufficient conditions for the existence of an optimal solution to the LSP. Bruss and Robertson (1988) extend these results to develop a dynamic program for any discrete distribution of the object's location.

More closely related to our treatment of the pinging period problem, Foley et al. (1991) consider a variant of the LSP in which the searcher can only travel a fixed, finite distance. The objective is to determine an optimal search plan that maximizes the probability of detecting the object while satisfying the distance constraint. The authors prove that there always exists a simple optimal search plan—one characterized by only one turning point—and provide closed form solutions for distributions including the standard normal, Cauchy, triangular and uniform. This bounded resource LSP is generalized as a rendezvous game in Alpern and Beck (1997, 1999). Washburn (1995) uses dynamic programming to solve a variant of the LSP in which the searcher has the option of carrying or leaving a heavy backpack during the search for a fixed marker. Demaine et al. (2006) explore another variation of the LSP in which the searcher incurs a cost for each change of direction.

A novelty of our model is that the hidden object is intermittently detectable while the searcher moves in continuous time and space, whereas the existing literature is often restricted to discrete search opportunities within a discretized search space. For example, Kadane (1968) develops strategies for finding an item hidden in a collection of boxes when there is a probability of overlooking the item and Nakai (1981) considers a similar formulation with time-dependent probabilities of detection. Onaga (1971) considers a continuous-time search over a discrete search space with fixed costs of switching to another search area. The author constructs search policies with the objective of either maximizing the probability of detection over a finite time horizon or minimizing the average time to detection.

To our knowledge, this paper is the first to explore a major design consideration of underwater locator beacons—the pinging period—and its impact on the probability of a successful search. We explore the design problem faced by aviation regulators and beacon manufacturers, namely selecting the pinging period before other parameters of the search, such as the search vessel speed, are known.

We assume that the search speed is selected *a priori* from a known distribution that accounts for factors such as ocean current and boat technology. This assumption differs from past efforts in which the search speed is either a known constant (e.g., Gal (1979)) or is variable over the course of the search (e.g., Alpern and Lidbetter (2015)). In searches for FDRs, the search speed depends on the depth and surface conditions of the search region, as well as the operating speeds of search vessels. Furthermore, the search vessel's speed is a major limitation on the search effort, rather than a variable that could significantly improve the likelihood of locating the FDR.

We apply the definite range law, under which the probability of detection is one if the search vessel is within a given range of the beacon when a ping occurs, and zero otherwise (see for example Koopman (1980)). Although the definite range law may not always apply in practice, it allows us to derive analytical expressions for the probability of a successful search. As in Foley et al. (1991), we assume that there is only one search vessel and that once the beacon has been detected, the search terminates. Our approach differs from Foley et al. (1991), however, in that the searcher does not need to be at the exact location of the beacon, but merely within range of it at an instant in which it is detectable.

The remainder of the paper is organized as follows. In Section 2, we formulate a mathematical model of the search problem with limited, periodic detection opportunities and introduce relevant notation. Section 3 examines linear search trajectories with constant search speeds over a finite-length interval; we derive the probability of detection when the prior distribution of the beacon's location is uniform. Section 4 generalizes the model to uncertain search speeds. Section 5 applies these results to current beacon design and discusses its implications on modern-day search efforts, in particular the search for Malaysian Airlines Flight 370. Section 6 summarizes our major conclusions and presents open topics for future research.

2 Model Formulation

Let (X, d) be a metric space and let $Q \subset X$ be a compact search space on which a searcher seeks an immobile beacon located at some unknown point $B \in Q$. Every τ units of time, the beacon instantaneously transmits a ping that is audible only within a radius r_b of point B. Similarly, the searcher can detect pings within a radius r_s of the searcher's location. When a ping is transmitted and these ranges overlap, the searcher detects the beacon's ping and the search terminates. An equivalent formulation is one in which the searcher detects the beacon when within a radius $r := r_b + r_s$ of the beacon's location, B, and a ping is transmitted.

Assume the beacon transmits a total of $n+1$ pings with the initial ping occurring at time $t=0$. Hence, the maximum duration of the search is $T := n\tau$ time units after which the beacon is undetectable. Let $S : \mathbb{R}_+ \mapsto X$ be a continuous function describing the location of the searcher at time t, and assume the searcher travels at a constant speed ν. Let Ω_i^S be the set of points in X being searched at the instant of the i^{th} ping, $i = 0, \ldots, n$, when following the trajectory described by S, i.e., $\Omega_i^S = \{x \in X : d(x, S(i\tau)) \leqslant r\}$. Furthermore, let $\Omega^S(k\tau) = \bigcup_{i=0}^k \Omega_i^S$, $k = 0, \ldots, n$, be the cumulative set of points searched up to time $k\tau$. For a fixed search trajectory S, we omit S from Ω_i^S and $\Omega^S(k\tau)$ for notational convenience.

Let F be a proper probability distribution over Q describing the searcher's prior belief on the beacon's location B. For any subset $A \subset Q$, let $\mu(A) = \Pr(B \in A)$ be the measure of the subset with respect to F, in which case the probability that the searcher detects the beacon before time $k\tau$ is $\mu(\Omega^S(k\tau))$. For a fixed search speed ν and fixed F, the probability $\mu(\Omega^S(k\tau))$ is a function of k and the pinging period τ only. When ν is determined by a general distribution G, we redefine this detection probability to reflect the fact that it is a function of both the search speed and pinging period by letting $\mu(\Omega^S(k\tau)) =: \theta_k(\nu, \tau)$. When we consider $\theta_n(\nu, \tau)$, the probability of detecting the beacon before time T, we drop the subscript n for notational convenience.

Next, suppose that the searcher receives a non-negative reward γ_i when the beacon is detected at the instant of the i^{th} ping, $i = 0, \ldots, n$. Let $\Gamma(\tau)$ be the searcher's expected reward when the pinging period is τ, i.e.,

$$\Gamma(\tau) = \gamma_0 \theta_0(\nu, \tau) + \sum_{i=1}^{n} \gamma_i \left[\theta_i(\nu, \tau) - \theta_{i-1}(\nu, \tau)\right]. \tag{1}$$

Note that when $\gamma_i = 1$ for $i = 0, \ldots, n$, this expected reward is equivalent to the probability of detecting the beacon by time T, i.e., $\Gamma(\tau) = \theta(\nu, \tau)$.

In the remainder of the paper, we consider the case in which the search space is a real interval of finite length L; i.e., $Q = [0, L]$ for some $L > r$. We assume $X = \mathbb{R}$ is equipped with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Hence, Ω_i^S, $i = 0, \ldots, n$, is an interval of length $2r$ centered around the location $S(i\tau)$ and $\Omega^S(i\tau)$ is a finite union of $i+1$ intervals.

3 Fixed Search Speed

In this section, we consider the case in which the search speed assumes a fixed value ν and the objective is to determine the optimal pinging period, τ^*, which maximizes $\Gamma(\tau)$, the searcher's expected reward. We further assume that the searcher's starting location is the left endpoint of the search space, i.e., $S(0) = 0$. To simplify the results that follow, the searcher is assumed to follow a linear trajectory $S(t) = \nu t$ for $0 \leqslant t \leqslant T$.

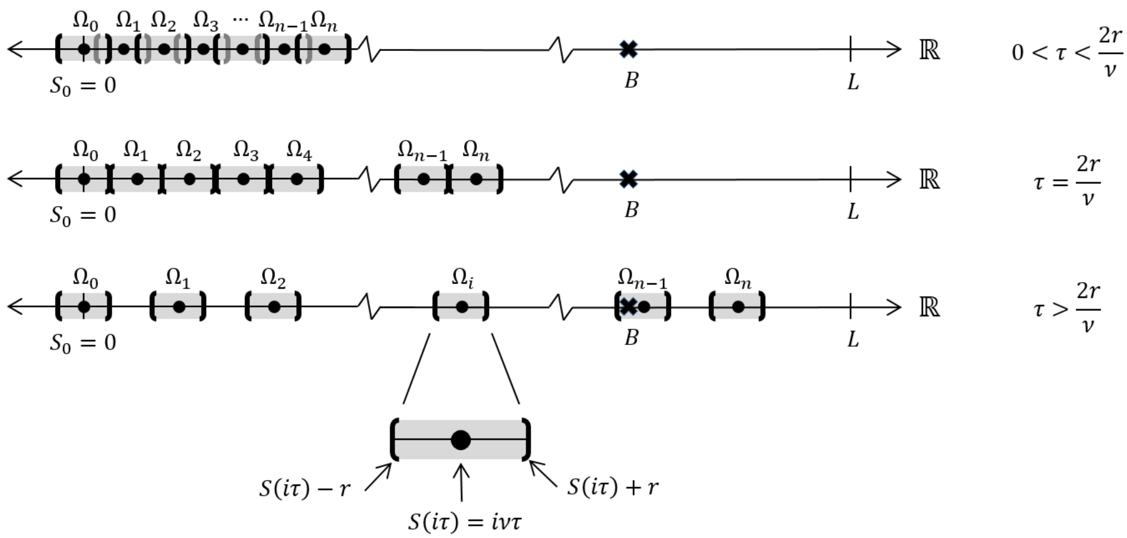


Figure 1: A search on the space $Q = [0,\, L]$ in which the searcher travels rightward from 0 under three settings of the pinging period τ. The searcher's position at time $i\tau$ and corresponding detection intervals Ω_i are represented by dots and gray shaded regions respectively. An arbitrary beacon location, B, is marked with an "X."

For the linear search trajectory $S(t) = \nu t$, the probability of detecting the beacon before time T is given by

$$\theta(\nu,\, \tau) = \mu(\Omega(n\tau)) = \mu\left(\bigcup_{i=0}^{n} \Omega_i\right) = \mu\left(\bigcup_{i=0}^{n} [S(i\tau) - r,\, S(i\tau) + r]\right).$$

As shown in Figure 1, for sufficiently short pinging periods, i.e., for values of $\tau < 2r/\nu$, the intervals searched during successive pings overlap, i.e., $\Omega_i \cap \Omega_{i-1} \neq \emptyset$ for $i = 1, \ldots, n$, thus $\Omega(n\tau) = [-r,\, n\nu\tau + r]$. Consequently,

$$\theta(\nu,\, \tau) = F(n\nu\tau + r) \quad \text{for } \tau < 2r/\nu. \tag{2}$$

It is evident from Figure 1 that when $\tau = 2r/\nu$, the searcher travels exactly twice the radius of detection between successive pings and therefore avoids overlapping searched intervals, but leaves no intervals unsearched as it progresses. For this pinging period, $\Omega(n\tau) = [-r,\, 2rn + r]$, which clearly is a superset of the interval $[-r,\, n\nu\tau + r]$ searched when $\tau < 2r/\nu$. Hence, the pinging period that maximizes $\theta(\nu,\, \tau)$ is no less than $2r/\nu$, i.e., $\tau^* \geqslant 2r/\nu$.

As shown in Figure 1, for sufficiently long pinging periods, i.e., for values of $\tau > 2r/\nu$, the searched intervals are disjoint, hence

$$\theta(\nu,\, \tau) = \sum_{i=0}^{n} \left[F(i\nu\tau + r) - F(i\nu\tau - r)\right] \quad \text{for } \tau > 2r/\nu. \tag{3}$$

Finally, for excessively long pinging periods, i.e., for values of $\tau > \frac{L+r}{\nu}$, only the interval searched at the time of the zeroth ping intersects $[0,\, L]$; thus $\theta(\nu,\, \tau) = F(r)$. Because the interval $[-r,\, r]$ is detected during the zeroth ping regardless of the pinging period, it follows that $\tau^* \leqslant \frac{L+r}{\nu}$. Therefore we restrict our search for the optimal pinging period, τ^*, to the interval $\left[\frac{2r}{\nu},\, \frac{L+r}{\nu}\right]$.

For an arbitrary F, determining the optimal pinging period may be analytically intractable given the form of $\theta(\nu,\, \tau)$ in (3). However, Proposition 1 establishes sufficient conditions on F and γ_i under which the pinging period $\tau = 2r/\nu$ maximizes the expected reward. It is interesting to note that this optimal pinging period does not depend on L, the length of the search space.

Proposition 1 *Let $F(b) = \Pr(B \leqslant b)$ be a distribution on $[0,\, L]$ with density $f(b)$. If $f(b)$ is nonincreasing on the interval $[r,\, L]$ and the sequence $\{\gamma_i\}_{i=0}^{n}$ is nonincreasing, then $\tau^* = 2r/\nu$.*

Proof. It follows from (1) and (2) that for any $\tau < 2r/\nu$,

$$\Gamma(\tau) = \gamma_0 F(r) + \sum_{i=1}^{n} \gamma_i \left[F(i\nu\tau + r) - F((i-1)\nu\tau + r)\right] = \sum_{i=0}^{n-1} F(i\nu\tau + r)\left[\gamma_i - \gamma_{i+1}\right] + \gamma_n F(n\nu\tau + r).$$

Differentiating with respect to τ,

$$\Gamma'(\tau) = \sum_{i=0}^{n-1} i\nu f(i\nu\tau + r)\left[\gamma_i - \gamma_{i+1}\right] + \gamma_n n\nu f(n\nu\tau + r) \geqslant 0.$$

Thus $\Gamma(\tau) \leqslant \Gamma(2r/\nu)$ for $\tau < 2r/\nu$.

It also follows from (1) and (3) that for any $\tau > 2r/\nu$,

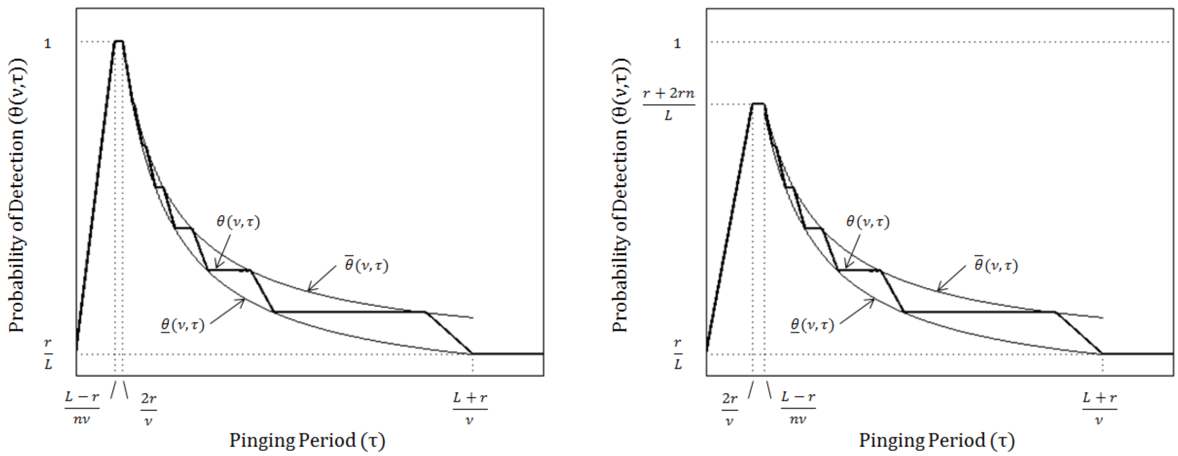
$$\begin{aligned}\Gamma(\tau) &= \sum_{i=0}^{n} \gamma_i \left[F(i\nu\tau + r) - F(i\nu\tau - r)\right] \\ &= \sum_{i=0}^{n} \gamma_i \int_{i\nu\tau - r}^{i\nu\tau + r} f(b)\,db \\ &\leqslant \sum_{i=0}^{n} \gamma_i \int_{2ri - r}^{2ri + r} f(b)\,db \\ &= \Gamma(2r/\nu). \quad \square\end{aligned}$$

The intuition behind Proposition 1 is straightforward. As it becomes less likely that the beacon is located far from the searcher's starting location, longer periods between pings result in skipped intervals close to the origin that are more likely to contain the beacon than intervals of equal length located farther from the origin. Likewise, shorter periods between pings result in inefficient overlapping of searched intervals, thereby reducing the length of the total searched interval and, consequently, the probability of locating the beacon.

We remark that under the conditions of Proposition 1, our assumption that the initial ping occurs at time $t = 0$ is a mild one. If instead the initial ping occurred at time t_0 with subsequent pings at times $t_0 + i\tau$, $i = 1, \ldots, n$, the intervals Ω_i^S are merely shifted νt_0 units to the right, and the result still holds.

Throughout the remainder of the paper we assume $\gamma_i = 1$ for $i = 0, \ldots, n$ and a uniform beacon location distribution, i.e, $F(b) = b/L$ for $b \in [0, L]$, which together satisfy the conditions of Proposition 1. For the uniform location distribution, the probability of detection is equal to the proportion of the interval $[0, L]$ searched up to time T. In Figure 2, we describe and plot the probability of detection, $\theta(\nu, \tau)$, as a function of the pinging period, τ, for the following two cases:

Sufficient Pings. We refer to the case in which the beacon pings often enough that the searcher is able to search all of $[0, L]$ before time T as "sufficient pings." This condition is satisfied when the interval $[r, L]$ can be covered by n intervals of length $2r$, i.e., $n \geqslant \frac{L-r}{2r}$. In this case, $\theta(\nu, \tau)$ is



(a) Sufficient Pings (b) Insufficient Pings

Figure 2: The probability of detection $\theta(\nu,\tau)$ as a function of the pinging period τ assuming a uniform location distribution and fixed search speed ν. Lower and upper bounds $\underline{\theta}(\nu,\tau)$ and $\overline{\theta}(\nu,\tau)$, respectively.

given by the piecewise linear function shown on Figure 2(a), namely

$$\theta(\nu,\tau)=\begin{cases}\dfrac{r+\nu\tau n}{L} & \text{if } 0<\tau<\frac{L-r}{n\nu}, \quad \text{(4a)}\\ 1 & \text{if } \frac{L-r}{n\nu}\leqslant\tau\leqslant\frac{2r}{\nu}, \quad \text{(4b)}\\ \dfrac{1}{L}\left(r+2r\left\lfloor\dfrac{L-r}{\nu\tau}\right\rfloor+\left[L+r-\nu\tau\left(\left\lfloor\dfrac{L-r}{\nu\tau}\right\rfloor+1\right)\right]^{+}\right) & \text{if } \frac{2r}{\nu}<\tau\leqslant\frac{L+r}{\nu}, \quad \text{(4c)}\\ \dfrac{r}{L} & \text{if } \tau>\frac{L+r}{\nu}. \quad \text{(4d)}\end{cases}$$

For pinging periods in the range of (4a), the searcher searches the interval $[-r,\,\nu\tau n+r]$ and thus $\theta(\nu,\tau)$ is linearly increasing in τ. Within the range of (4b), the entire interval $[0,\,L]$ is searched and thus $\theta(\nu,\tau)$ is constant and attains its maximum value of one. For (4c), the searcher leaves intervals unsearched, i.e., the intervals Ω_i are disjoint. The breakpoints of the maximum expression in (4c) occur when the point L is either the left or right endpoint of a searched interval Ω_i. Values of τ which satisfy this condition are of the form $\frac{L-r}{i\nu}$ for $i=1,\ldots,\left\lfloor\frac{L-r}{2r}\right\rfloor$ and $\frac{L+r}{i\nu}$ for $i=1,\ldots,\left\lfloor\frac{L-r}{2r}\right\rfloor+1$. The function $\theta(\nu,\tau)$ is constant on ranges of τ for which $L\notin\Omega_i$ for all i, while it is linearly decreasing for ranges of τ for which one of the Ω_i contains the point L. Finally, when the pinging period is in the range of (4d), only Ω_0 intersects $[0,\,L]$.

Insufficient Pings. We refer to the more interesting case in which there are not enough pings for the searcher to search all of $[0, L]$ before time T, i.e., when $n < \frac{L-r}{2r}$, as "insufficient pings." In this case, the probability of detection is strictly less than one for all pinging periods. The probability of detection, $\theta(\nu, \tau)$, is given by the piecewise linear function shown in Figure 2(b), namely

$$\theta(\nu, \tau) = \begin{cases} \dfrac{r + \nu\tau n}{L} & \text{if } 0 < \tau < \frac{2r}{\nu}, \qquad \text{(5a)} \\ \dfrac{r + 2rn}{L} & \text{if } \frac{2r}{\nu} \leqslant \tau \leqslant \frac{L-r}{n\nu}, \qquad \text{(5b)} \\ \dfrac{1}{L}\left(r + 2r\left\lfloor \dfrac{L-r}{\nu\tau} \right\rfloor + \left[L + r - \nu\tau\left(\left\lfloor \dfrac{L-r}{\nu\tau}\right\rfloor + 1\right)\right]^+\right) & \text{if } \frac{L-r}{n\nu} < \tau \leqslant \frac{L+r}{\nu}, \qquad \text{(5c)} \\ \dfrac{r}{L} & \text{if } \tau > \frac{L+r}{\nu}. \qquad \text{(5d)} \end{cases}$$

The function given by (5a)–(5d) has a few notable differences from that given by (4a)–(4d). Note that in (5b), $\theta(\nu, \tau) < 1$ since the searcher is unable to search all of $[0, L]$. The breakpoints of the maximum expression in (5c) are values of τ of the form $\frac{L-r}{i\nu}$ for $i = 1, \ldots, n-1$ and $\frac{L+r}{i\nu}$ for $i = 1, \ldots, n$, which differ from the breakpoints of (4c). However, it can be easily seen that $\tau^* = 2r/\nu$ maximizes $\theta(\nu, \tau)$ in both (4) and (5).

Because the expressions in (4c) and (5c) are analytically cumbersome, we introduce simple lower and upper bounds that prove useful in approximating $\mathbb{E}_\nu[\theta(\nu, \tau)]$ in Section 4. These bounds, $\underline{\theta}(\nu, \tau)$ and $\overline{\theta}(\nu, \tau)$, respectively, are derived in Proposition 2 and depicted in Figure 2 on the intervals given in (4c) and (5c).

Proposition 2 *When there are sufficient pings* $\left(n \geqslant \frac{L-r}{2r}\right)$ *and* $\tau \in \left[\frac{2r}{\nu}, \frac{L+r}{\nu}\right]$, *or when there are insufficient pings* $\left(n < \frac{L-r}{2r}\right)$ *and* $\tau \in \left[\frac{L-r}{n\nu}, \frac{L+r}{\nu}\right]$, $\theta(\nu, \tau)$ *is bounded below and above by the monotonically decreasing functions* $\underline{\theta}(\nu, \tau)$ *and* $\overline{\theta}(\nu, \tau)$, *defined as*

$$\underline{\theta}(\nu, \tau) := \frac{2r}{L}\left(\frac{L+r}{\nu\tau}\right) - \frac{r}{L} \leqslant \theta(\nu, \tau) \leqslant \frac{2r}{L}\left(\frac{L-r}{\nu\tau}\right) + \frac{r}{L} =: \overline{\theta}(\nu, \tau).$$

In the case of sufficient pings,

$$\underline{\theta}\left(\nu, \frac{2r}{\nu}\right) = \theta\left(\nu, \frac{2r}{\nu}\right) = \overline{\theta}\left(\nu, \frac{2r}{\nu}\right) = 1, \qquad (6)$$

and in the case of insufficient pings,

$$\underline{\theta}\left(\nu,\ \frac{L-r}{n\nu}\right) < \theta\left(\nu,\ \frac{L-r}{n\nu}\right) = \overline{\theta}\left(\nu,\ \frac{L-r}{n\nu}\right) = \frac{r+2rn}{L}. \tag{7}$$

Proof. Equations (6) and (7) follow from the definitions of $\underline{\theta}(\nu,\,\tau)$ and $\overline{\theta}(\nu,\,\tau)$.

For a fixed search speed ν and pinging period τ, let $\rho(L)$ be the total length of the searched area $\Omega(n\tau)$ that falls within the search space $[0, L]$. For an instance in which the search intervals do not overlap, or only overlap trivially, (i.e., $\nu\tau \geqslant 2r$), the function $\rho(L)$ is given by

$$\rho(L) = \begin{cases} L & \text{if } 0 < L \leqslant r, \\ (2i-1)r & \text{if } (i-1)\nu\tau + r < L \leqslant i\nu\tau - r \text{ for } i = 1,\ldots,n, \\ (2i-1)r + L - (i\nu\tau - r) & \text{if } i\nu\tau - r < L \leqslant i\nu\tau + r \text{ for } i = 1,\ldots,n, \\ (2n+1)r & \text{if } L > n\nu\tau + r, \end{cases}$$

and is shown in Figure 3. Observe that $\rho(L)$ is nondecreasing since as L increases, more of the searched intervals fall within the search space.

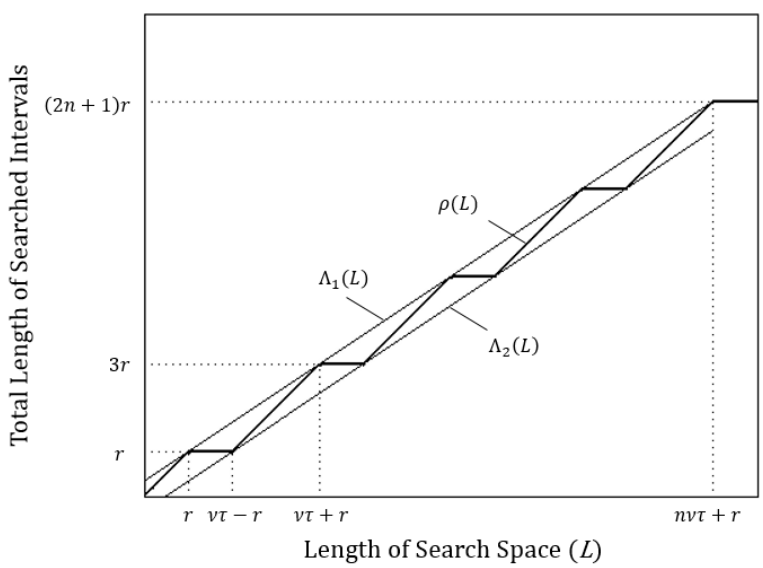


Figure 3: The total measure (length) of search points, $\rho(L)$, as a function of the length of the search space for fixed search speed ν and pinging period τ satisfying $\nu\tau > 2r$ with upper and lower bounds $\Lambda_1(L)$ and $\Lambda_2(L)$.

Consider the linear functions

$$\Lambda_1(L) = \frac{2r}{\nu\tau}L + r - \frac{2r^2}{\nu\tau} \quad \text{and} \quad \Lambda_2(L) = \frac{2r}{\nu\tau}L - r + \frac{2r^2}{\nu\tau}.$$

It can be easily verified that for $\nu\tau - r \leqslant L \leqslant n\nu\tau + r$, $\rho(L)$ is bounded above and below by $\Lambda_1(L)$ and $\Lambda_2(L)$, respectively. The plot of $\rho(L)$ in Figure 3 illustrates the validity of these bounds.

By fixing the length of the search space, the probability of successful detection is equal to the total measure (length) of the searched points divided by the length of the search space, i.e., $\theta(\nu, \tau) = \rho(L)/L$. Hence $\Lambda_2(L)/L \leqslant \theta(\nu, \tau) \leqslant \Lambda_1(L)/L$. A straightforward derivation shows that $\overline{\theta}(\nu, \tau) = \Lambda_1(L)/L$ and $\underline{\theta}(\nu, \tau) = \Lambda_2(L)/L$ and therefore the bounds of $\theta(\nu, \tau)$ hold for the cases of sufficient and insufficient pings. $\square$

4 Discrete Search Speed Distribution

In this section, we consider the case in which the search speed ν is not fixed, which may occur for example, if the searcher's vessel type is uncertain, or if ocean current conditions hinder the progress of the vessel. To be more precise, we assume that the search speed is not known before the pinging frequency must be determined, but that the search speed is fixed for any particular search. We consider the case when the distribution of search speeds, G, is discrete over speeds $\nu_1 < \nu_2 < \cdots < \nu_m$, m finite. Let $g(\nu_j) = \Pr(\nu = \nu_j)$ for $j = 1, \ldots, m$, $\sum_{j=1}^{m} g(\nu_j) = 1$, and $\mathbb{E}[\nu] = \sum_{j=1}^{m} \nu_j g(\nu_j)$. Our objective is to find the pinging period τ^* that maximizes the expected probability of detecting the beacon, i.e., $\mathbb{E}_\nu[\theta(\nu, \tau)] = \sum_{j=1}^{m} \theta(\nu_j, \tau) g(\nu_j)$.

4.1 Two Search Speeds

Suppose that there are two possible search speeds with probabilities $g(\nu_1) = \lambda$ and $g(\nu_2) = 1 - \lambda$, $0 < \lambda < 1$, so $\mathbb{E}_\nu[\theta(\nu, \tau)] = \lambda\theta(\nu_1, \tau) + (1 - \lambda)\theta(\nu_2, \tau)$. In many of the results that follow (Lemma 1, Theorem 1, and Corollaries 1–3), we assume there are sufficient pings, i.e., $n \geqslant \frac{L-r}{2r}$.

If $\nu_1 \geqslant \frac{L-r}{2rn}\nu_2$, then $\theta(\nu_1, \tau) = \theta(\nu_2, \tau) = 1$ for values of τ on the nonempty interval $\left[\frac{L-r}{n\nu_1}, \frac{2r}{\nu_2}\right]$. Hence selecting any $\tau \in \left[\frac{L-r}{n\nu_1}, \frac{2r}{\nu_2}\right]$ achieves the optimal value of $\mathbb{E}_\nu[\theta(\nu, \tau)] = 1$. Suppose instead that $\nu_1 < \frac{L-r}{2rn}\nu_2$, implying that there is a proper interval of pinging periods $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$ on which at most one of the functions $\theta(\nu_1, \tau)$ and $\theta(\nu_2, \tau)$ achieves its maximum. The functions $\theta(\nu_1, \tau)$ and $\theta(\nu_2, \tau)$ are both nondecreasing in τ for $\tau < 2r/\nu_2$ and nonincreasing for $\tau > \frac{L-r}{n\nu_1}$, as shown in Figure 4. As a convex combination of these two functions, $\mathbb{E}_\nu[\theta(\nu, \tau)]$ is nondecreasing and nonincreasing, respectively, on these intervals. Thus $\frac{2r}{\nu_2} \leqslant \tau^* \leqslant \frac{L-r}{n\nu_1}$. Lemma 1 constructs a linear upper bound on $\theta(\nu_2, \tau)$ on this interval, with equality holding at the endpoints.

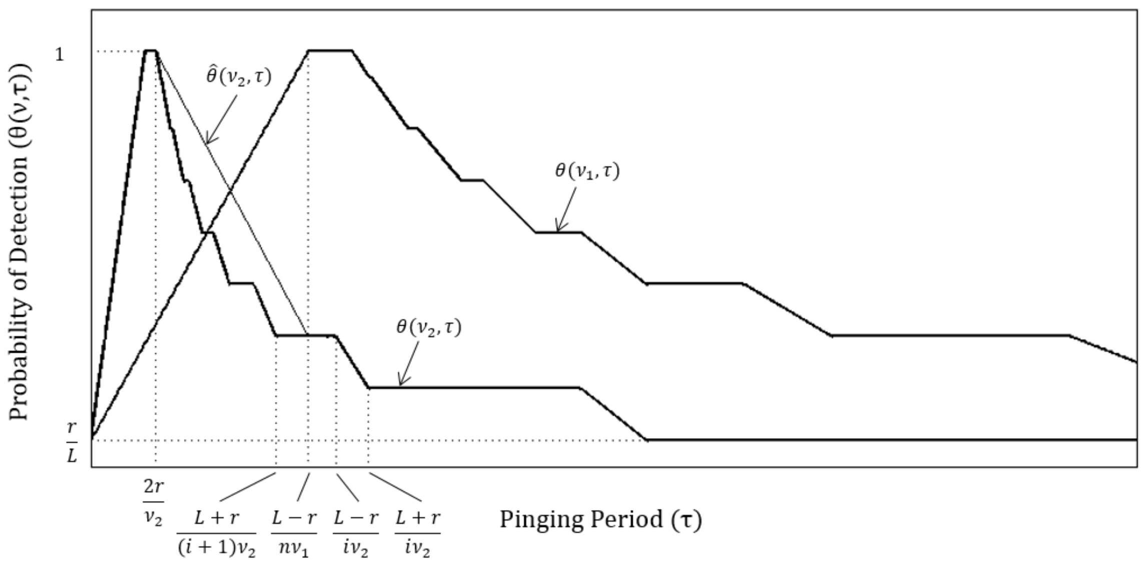


Figure 4: The linear upper bound $\hat{\theta}(\nu_2, \tau)$ of the probability of detection under the faster search speed ν_2 applied to the interval of pinging periods $\tau \in \left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$.

Lemma 1 *For a fixed search speed* ν*, sufficient pings* $\left(n \geqslant \frac{L-r}{2r}\right)$*, and* $\tau' > \frac{2r}{\nu}$*, the probability of detection* $\theta(\nu, \tau)$ *is bounded above on the interval of pinging periods* $\left[\frac{2r}{\nu}, \tau'\right]$ *by the line segment* $\hat{\theta}(\nu, \tau)$ *connecting* $\left(\frac{2r}{\nu}, \theta\left(\nu, \frac{2r}{\nu}\right)\right)$ *and* $(\tau', \theta(\nu, \tau'))$.

Proof. An equivalent statement of Lemma 1 is that for any pinging period $\tilde{\tau} \geqslant 2r/\nu$, the bound $\theta(\nu, \tilde{\tau}) \leqslant \hat{\theta}(\nu, \tilde{\tau})$ holds for all $\tau' \geqslant \tilde{\tau}$ where again $\hat{\theta}(\nu, \tilde{\tau})$ is dependent on τ'. The bound clearly holds for $\tilde{\tau} = 2r/\nu$ since from the definition of $\hat{\theta}(\nu, \tau)$, $\theta(\nu, \tilde{\tau}) = \theta\left(\nu, \frac{2r}{\nu}\right) = \hat{\theta}\left(\nu, \frac{2r}{\nu}\right) = \hat{\theta}(\nu, \tilde{\tau})$, for all $\tau' \geqslant 2r/\nu$.

For fixed $\tilde{\tau} > 2r/\nu$, $\hat{\theta}(\nu, \tilde{\tau}) = \theta(\nu, \tilde{\tau})$ when $\tau' = \tilde{\tau}$ and $\hat{\theta}(\nu, \tau)$ is a straight line for any $\tau' > \tilde{\tau}$. It suffices to show that the gradient of $\hat{\theta}(\nu, \tau)$ with respect to τ is non-decreasing in τ'. Then because $\hat{\theta}(\nu, \tau)$ is anchored at the point $\left(\frac{2r}{\nu}, \theta\left(\nu, \frac{2r}{\nu}\right)\right)$, it follows that as τ' increases, $\hat{\theta}(\nu, \tilde{\tau})$ will not decrease.

Observe that the line segment $\hat{\theta}(\nu,\tau)$ can be written as

$$\hat{\theta}(\nu,\,\tau) = \theta\left(\nu,\,\frac{2r}{\nu}\right) - \frac{\theta(\nu,\,\frac{2r}{\nu}) - \theta(\nu,\,\tau')}{\tau' - \frac{2r}{\nu}}\left(\tau - \frac{2r}{\nu}\right).$$

Because there are sufficient pings, by (4b), $\theta(\nu,\,\frac{2r}{\nu}) = 1$ and so

$$\hat{\theta}(\nu,\,\tau) = 1 - \frac{1 - \theta(\nu,\,\tau')}{\tau' - \frac{2r}{\nu}}\left(\tau - \frac{2r}{\nu}\right). \tag{8}$$

Differentiating (8) with respect to τ yields

$$\frac{\partial\hat{\theta}(\nu,\,\tau)}{\partial\tau} = \frac{\theta(\nu,\tau') - 1}{\tau' - \frac{2r}{\nu}}. \tag{9}$$

Note that (9) is continuous in τ' because $\theta(\nu,\tau')$ is continuous in τ' and the denominator is strictly positive. To show that (9) is nondecreasing in τ', we differentiate with respect to τ' and show that its gradient is nonnegative where it is defined. By the quotient rule,

$$\frac{\partial}{\partial\tau'}\left(\frac{\theta(\nu,\,\tau') - 1}{\tau' - \frac{2r}{\nu}}\right) = \frac{\left(\tau' - \frac{2r}{\nu}\right)\left(\frac{\partial}{\partial\tau'}\theta(\nu,\,\tau')\right) - (\theta(\nu,\,\tau') - 1)}{\left(\tau' - \frac{2r}{\nu}\right)^2}. \tag{10}$$

Because the denominator of (10) is strictly positive, it remains to show that the numerator is nonnegative for $\tau' \geqslant \tilde{\tau}$. We achieve this by showing that the numerator of (10) is in fact nonnegative for all $\tau' > 2r/\nu$.

Equations (4c) and (4d), which express $\theta(\nu,\,\tau')$ in the case of sufficient pings for pinging periods $\tau' > 2r$, can be decomposed as

$$\theta(\nu,\tau') = \begin{cases} 1 - \frac{\nu}{L}\left(\left\lfloor\frac{L-r}{\nu\tau'}\right\rfloor + 1\right)\left(\tau' - \frac{2r}{\nu}\right) & \text{if } \tau' \in \left(\frac{2r}{\nu},\, \frac{L+r}{\nu\left(\left\lfloor\frac{L-r}{2r}\right\rfloor+1\right)}\right], & \text{(11a)} \\ 1 - \frac{\nu}{L}\left(\left\lfloor\frac{L-r}{\nu\tau'}\right\rfloor\right)\left(\frac{L-r}{i\nu} - \frac{2r}{\nu}\right) & \text{if } \tau' \in \left(\frac{L+r}{(i+1)\nu},\, \frac{L-r}{i\nu}\right] \text{ for } i = 1,\ldots,\left\lfloor\frac{L-r}{2r}\right\rfloor, & \text{(11b)} \\ 1 - \frac{\nu}{L}\left(\left\lfloor\frac{L-r}{\nu\tau'}\right\rfloor + 1\right)\left(\tau' - \frac{2r}{\nu}\right) & \text{if } \tau' \in \left(\frac{L-r}{i\nu},\, \frac{L+r}{i\nu}\right] \text{ for } i = 1,\ldots,\left\lfloor\frac{L-r}{2r}\right\rfloor, & \text{(11c)} \\ \frac{r}{L} & \text{if } \tau' > \frac{L+r}{\nu}. & \text{(11d)} \end{cases}$$

We proceed in cases on $\tau' > 2r/\nu$.

Case 1: Suppose that $\tau' \in \left(\frac{2r}{\nu}, \frac{L+r}{\nu\left(\left\lfloor \frac{L-r}{2r} \right\rfloor + 1\right)}\right]$. From (11a),

$$\left(\tau' - \frac{2r}{\nu}\right)\left(\frac{\partial}{\partial \tau'}\theta(\nu, \tau')\right) - (\theta(\nu, \tau') - 1)$$
$$= \left(\tau' - \frac{2r}{\nu}\right)\left[-\frac{\nu}{L}\left(\left\lfloor \frac{L-r}{\nu\tau'} \right\rfloor + 1\right)\right] + \left[\frac{\nu}{L}\left(\left\lfloor \frac{L-r}{\nu\tau'} \right\rfloor + 1\right)\left(\tau' - \frac{2r}{\nu}\right)\right] = 0.$$

Case 2: Suppose that $\tau' \in \left(\frac{L+r}{(i+1)\nu}, \frac{L-r}{i\nu}\right]$ for some $i \in \{1, \ldots, \lfloor \frac{L-r}{2r} \rfloor\}$. From (11b),

$$\left(\tau' - \frac{2r}{\nu}\right)\left(\frac{\partial}{\partial \tau'}\theta(\nu, \tau')\right) - (\theta(\nu, \tau') - 1)$$
$$= \left(\tau' - \frac{2r}{\nu}\right)(0) + \left[\frac{\nu}{L}\left(\left\lfloor \frac{L-r}{\nu\tau'} \right\rfloor\right)\left(\frac{L-r}{i\nu} - \frac{2r}{\nu}\right)\right] \geqslant 0.$$

Case 3: Suppose that $\tau' \in \left(\frac{L-r}{i\nu}, \frac{L+r}{i\nu}\right]$ for some $i \in \{1, \ldots, \lfloor \frac{L-r}{2r} \rfloor\}$. From (11c),

$$\left(\tau' - \frac{2r}{\nu}\right)\left(\frac{\partial}{\partial \tau'}\theta(\nu, \tau')\right) - (\theta(\nu, \tau') - 1)$$
$$= \left(\tau' - \frac{2r}{\nu}\right)\left[-\frac{\nu}{L}\left(\left\lfloor \frac{L-r}{\nu\tau'} \right\rfloor + 1\right)\right] + \left[\frac{\nu}{L}\left(\left\lfloor \frac{L-r}{\nu\tau'} \right\rfloor + 1\right)\left(\tau' - \frac{2r}{\nu}\right)\right] = 0.$$

Case 4: Suppose that $\tau' > \frac{L+r}{\nu}$. From (11d),

$$\left(\tau' - \frac{2r}{\nu}\right)\left(\frac{\partial}{\partial \tau'}\theta(\nu, \tau')\right) - (\theta(\nu, \tau') - 1) = \left(\tau' - \frac{2r}{\nu}\right)(0) - \left(\frac{r}{L} - 1\right) = 1 - \frac{r}{L} \geqslant 0.$$

From these cases, we have shown that (10) is nonnegative and hence (9) is nondecreasing in τ'. Therefore the inequality $\theta(\nu, \tilde{\tau}) \leqslant \hat{\theta}(\nu, \tilde{\tau})$ is satisfied for all $\tau' \geqslant \tilde{\tau}$. □

The linear upper bound of Lemma 1, applied to $\theta(\nu_2, \tau)$, is now used in Theorem 1 to construct a linear upper bound for $\mathbb{E}_\nu[\theta(\nu, \tau)]$ on the interval $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$ with equality at the endpoints. We then show that the probabilities of search speeds ν_1 and ν_2 explicitly determine τ^*.

Theorem 1 *If there are sufficient pings, there exists a* $\tilde{\lambda}$ *such that for* $\lambda \leqslant \tilde{\lambda}$, $\tau^* = 2r/\nu_2$ *and for* $\lambda \geqslant \tilde{\lambda}$, $\tau^* = \frac{L-r}{n\nu_1}$.

Proof. Applying the linear upper bound $\hat{\theta}(\nu_2, \tau)$ of Lemma 1 on the interval $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$,

$$\mathbb{E}_\nu[\theta(\nu, \tau)] \leqslant \lambda\theta(\nu_1, \tau) + (1-\lambda)\hat{\theta}(\nu_2, \tau) =: \widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)].$$

Recall from (4a) that $\theta(\nu_1, \tau)$ is linearly increasing up to $\tau = \frac{L-r}{n\nu_1}$. As the convex combination of two linear functions, $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ is linear on the interval $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$. Let $\tilde{\lambda}$ be the parameter that makes $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ constant on the interval $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$, by satisfying the condition:

$$\widehat{\mathbb{E}}_\nu\left[\theta\left(\nu, \frac{2r}{\nu_2}\right)\right] = \widehat{\mathbb{E}}_\nu\left[\theta\left(\nu, \frac{L-r}{n\nu_1}\right)\right],$$

which can be written as

$$\tilde{\lambda}\theta\left(\nu_1, \frac{2r}{\nu_2}\right) + (1-\tilde{\lambda})\hat{\theta}\left(\nu_2, \frac{2r}{\nu_2}\right) = \tilde{\lambda}\theta\left(\nu_1, \frac{L-r}{n\nu_1}\right) + (1-\tilde{\lambda})\hat{\theta}\left(\nu_2, \frac{L-r}{n\nu_1}\right). \tag{12}$$

Because $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)] = \mathbb{E}_\nu[\theta(\nu, \tau)]$ for $\tau = 2r/\nu_2$ and $\tau = \frac{L-r}{n\nu_1}$, these two values of τ maximize $\mathbb{E}_\nu[\theta(\nu, \tau)]$ when $\lambda = \tilde{\lambda}$.

When $\lambda < \tilde{\lambda}$,

$$\begin{aligned}
&\widehat{\mathbb{E}}_\nu\left[\theta\left(\nu, \frac{2r}{\nu_2}\right)\right] - \widehat{\mathbb{E}}_\nu\left[\theta\left(\nu, \frac{L-r}{n\nu_1}\right)\right] \\
&= \lambda\left[\theta\left(\nu_1, \frac{2r}{\nu_2}\right) - \theta\left(\nu_1, \frac{L-r}{n\nu_1}\right)\right] + (1-\lambda)\left[\hat{\theta}\left(\nu_2, \frac{2r}{\nu_2}\right) - \hat{\theta}\left(\nu_2, \frac{L-r}{n\nu_1}\right)\right] \\
&\geqslant \tilde{\lambda}\left[\theta\left(\nu_1, \frac{2r}{\nu_2}\right) - \theta\left(\nu_1, \frac{L-r}{n\nu_1}\right)\right] + (1-\tilde{\lambda})\left[\hat{\theta}\left(\nu_2, \frac{2r}{\nu_2}\right) - \hat{\theta}\left(\nu_2, \frac{L-r}{n\nu_1}\right)\right] \qquad (13) \\
&= 0, \qquad (14)
\end{aligned}$$

where (13) is a result of $\theta(\nu_1, \tau)$ being linearly increasing and $\hat{\theta}(\nu_2, \tau)$ being linearly decreasing on the interval $\left[\frac{2r}{\nu_2}, \frac{L-r}{n\nu_1}\right]$ and (14) follows from the definition of $\tilde{\lambda}$. Therefore $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ attains its maximum at the left endpoint $\tau = 2r/\nu_2$. Because $\widehat{\mathbb{E}}_\nu[\theta(\nu, \frac{2r}{\nu_2})] = \mathbb{E}_\nu[\theta(\nu, \frac{2r}{\nu_2})]$, $\tau^* = 2r/\nu_2$.

A similar argument shows that when $\lambda > \tilde{\lambda}$, $\tau^* = \frac{L-r}{n\nu_1}$. $\square$

Corollary 1 proves that the value of $\tilde{\lambda}$ is greater than or equal to 1/2.

Corollary 1 *When the faster search speed ν_2 is as or more likely than the slower search speed ν_1, the optimal pinging period is the longest period that ensures no intervals are left undetected between* *pings, i.e.,* $\tau^* = 2r/\nu_2$.

Proof. Applying the upper bound $\overline{\theta}(\nu_2,\, \tau)$ of Proposition 2 on the interval $\left[\frac{2r}{\nu_2},\, \frac{L-r}{n\nu_1}\right]$,

$$\mathbb{E}_\nu[\theta(\nu,\, \tau)] \leqslant \lambda\theta(\nu_1,\, \tau) + (1-\lambda)\overline{\theta}(\nu_2,\, \tau) =: \overline{\mathbb{E}}_\nu[\theta(\nu,\, \tau)].$$

On the interval $\left[\frac{2r}{\nu_2},\, \frac{L-r}{n\nu_1}\right]$, $\overline{\mathbb{E}}_\nu[\theta(\nu,\, \tau)]$ is the sum of a linear function and a convex function; hence, it too is convex. Its maximum on the interval $\left[\frac{2r}{\nu_2},\, \frac{L-r}{n\nu_1}\right]$ thus occurs at one of the endpoints. Therefore the left endpoint $\tau = 2r/\nu_2$ is optimal if

$$\overline{\mathbb{E}}_\nu\left[\theta\left(\nu,\, \frac{2r}{\nu_2}\right)\right] > \overline{\mathbb{E}}_\nu\left[\theta\left(\nu,\, \frac{L-r}{n\nu_1}\right)\right],$$
$$\lambda\left(\frac{r}{L} + \frac{2rn\nu_1}{\nu_2 L}\right) + 1 - \lambda > \lambda + (1-\lambda)\left(\frac{r}{L} + \frac{2rn\nu_1}{\nu_2 L}\right). \tag{15}$$

Equation (15) readily reduces to $\lambda < 1/2$. From Theorem 1, it follows that $\tilde{\lambda} \geqslant 1/2$. □

Note that the result of Corollary 1 does not depend on the relative values of ν_1 and ν_2. Corollary 2 expresses $\tilde{\lambda}$ while satisfying (12) for the different functional forms of $\theta\left(\nu_2,\, \frac{L-r}{n\nu_1}\right)$.

Corollary 2 *If there are sufficient pings and* $\nu_1 < \frac{L-r}{2rn}\nu_2$*, the threshold value* $\tilde{\lambda}$ *is given by*

$$\tilde{\lambda} = \begin{cases} \dfrac{\left(\left\lfloor\frac{L-r}{2r}\right\rfloor + 1\right)\left(\frac{\nu_2}{n\nu_1}(L-r) - 2r\right)}{\left(\left\lfloor\frac{L-r}{2r}\right\rfloor + 1\right)\left(\frac{\nu_2}{n\nu_1}(L-r) - 2r\right) + L - r - 2r\frac{n\nu_1}{\nu_2}} & \text{for } \frac{2r}{\nu_2} \leqslant \frac{L-r}{n\nu_1} \leqslant \frac{L+r}{\nu_2\left(\left\lfloor\frac{L-r}{2r}\right\rfloor+1\right)}, \quad \text{(16a)} \\[2ex] \dfrac{L-r-2ri}{2\left(L-r-r\left(i+\frac{n\nu_1}{\nu_2}\right)\right)} & \text{for } \frac{L+r}{(i+1)\nu_2} \leqslant \frac{L-r}{n\nu_1} \leqslant \frac{L-r}{i\nu_2},\ 1 \leqslant i \leqslant \left\lfloor\frac{L-r}{2r}\right\rfloor, \quad \text{(16b)} \\[2ex] \dfrac{i\left(\frac{\nu_2}{n\nu_1}(L-r) - 2r\right)}{i\left(\frac{\nu_2}{n\nu_1}(L-r) - 2r\right) + L - r - 2r\frac{n\nu_1}{\nu_2}} & \text{for } \frac{L-r}{i\nu_2} \leqslant \frac{L-r}{n\nu_1} \leqslant \frac{L+r}{i\nu_2},\ 1 \leqslant i \leqslant \left\lfloor\frac{L-r}{2r}\right\rfloor, \quad \text{(16c)} \\[2ex] \dfrac{L-r}{2\left(L-r-r\frac{n\nu_1}{\nu_2}\right)} & \text{for } \frac{L+r}{i\nu_2} \leqslant \frac{L-r}{n\nu_1} < \infty. \quad \text{(16d)} \end{cases}$$

Proof. Rearranging (12) to solve for $\tilde{\lambda}$,

$$\tilde{\lambda} = \frac{\hat{\theta}\left(\nu_2,\, \frac{2r}{\nu_2}\right) - \hat{\theta}\left(\nu_2,\, \frac{L-r}{n\nu_1}\right)}{\hat{\theta}\left(\nu_2,\, \frac{2r}{\nu_2}\right) - \hat{\theta}\left(\nu_2,\, \frac{L-r}{n\nu_1}\right) + \theta\left(\nu_1,\, \frac{L-r}{n\nu_1}\right) - \theta\left(\nu_1,\, \frac{2r}{\nu_2}\right)}.$$

We now substitute $\hat{\theta}\left(\nu_2,\, \frac{2r}{\nu_2}\right) = \theta\left(\nu_2,\, \frac{2r}{\nu_2}\right) = 1$, $\hat{\theta}\left(\nu_2,\, \frac{L-r}{n\nu_1}\right) = \theta\left(\nu_2,\, \frac{L-r}{n\nu_1}\right)$, $\theta\left(\nu_1,\, \frac{L-r}{n\nu_1}\right) = 1$, and $\theta\left(\nu_1,\, \frac{L-r}{n\nu_1}\right) = \frac{r}{L} + \frac{2r}{L}\frac{n\nu_1}{\nu_2}$ (from (4a)), and multiply the numerator and denominator by L to obtain

$$\tilde{\lambda} = \frac{L - \theta(\nu_2,\, \frac{L-r}{n\nu_1})L}{2L - \theta(\nu_2,\, \frac{L-r}{n\nu_1})L - r - 2rn\frac{\nu_1}{\nu_2}}. \tag{17}$$

Recall from (11a)–(11c) that in the case of sufficient pings and $\tau \in \left(\frac{2r}{\nu}, \frac{L+r}{\nu}\right]$,

$$\theta(\nu,\tau) = \begin{cases} 1 - \dfrac{\nu}{L}\left(\left\lfloor \dfrac{L-r}{\nu\tau} \right\rfloor + 1\right)\left(\tau - \dfrac{2r}{\nu}\right) & \text{if } \tau \in \left(\frac{2r}{\nu}, \frac{L+r}{\nu\left(\left\lfloor\frac{L-r}{2r}\right\rfloor+1\right)}\right], \quad (18a) \\ 1 - \dfrac{\nu}{L}\left(\left\lfloor \dfrac{L-r}{\nu\tau} \right\rfloor\right)\left(\dfrac{L-r}{i\nu} - \dfrac{2r}{\nu}\right) & \text{if } \tau \in \left(\frac{L+r}{(i+1)\nu}, \frac{L-r}{i\nu}\right] \text{ for } i = 1, \ldots, \left\lfloor\frac{L-r}{2r}\right\rfloor, \quad (18b) \\ 1 - \dfrac{\nu}{L}\left(\left\lfloor \dfrac{L-r}{\nu\tau} \right\rfloor + 1\right)\left(\tau - \dfrac{2r}{\nu}\right) & \text{if } \tau \in \left(\frac{L-r}{i\nu}, \frac{L+r}{i\nu}\right] \text{ for } i = 1, \ldots, \left\lfloor\frac{L-r}{2r}\right\rfloor, \quad (18c) \\ \dfrac{r}{L} & \text{if } \tau' > \frac{L+r}{\nu}. \quad (18d) \end{cases}$$

Formulas (16a)–(16d) are then the result of evaluating (18a)–(18d) at $\tau = \frac{L-r}{n\nu_1}$ and substituting into (17). □

Lastly, Corollary 3 shows that it is possible to characterize the instances when $\tilde{\lambda} = 1/2$.

Corollary 3 *If there are sufficient pings and $\nu_1 < \frac{L-r}{2rn}\nu_2$, the threshold value $\tilde{\lambda}$ is equal to $1/2$ if and only if the product of the number of pings and the slower speed is a multiple of the faster speed, i.e., $n\nu_1 = i\nu_2$ for some $i \in \left\{1, \ldots, \left\lfloor\frac{L-r}{2r}\right\rfloor\right\}$.*

Proof. ($\Rightarrow$) Suppose that $\tilde{\lambda} = 1/2$. For this hold in (16c), it would need to be the case that

$$i\left(\frac{\nu_2}{n\nu_1}(L-r) - 2r\right) = L - r - 2r\frac{n\nu_1}{\nu_2},$$

which can be rearranged to reach the condition $\nu_1 = \frac{L-r}{2rn}\nu_2$ which cannot be satisfied because of the assumption that $\nu_1 < \frac{L-r}{2rn}\nu_2$. By setting $i = \left\lfloor\frac{L-r}{2r}\right\rfloor + 1$, it follows that $\tilde{\lambda} = 1/2$ is also impossible in (16a). From the expression (16d), it is evident from the fact that $r\frac{n\nu_1}{\nu_2} > 0$ that $\tilde{\lambda} > 1/2$ in this case. In the remaining case for (16b), $\tilde{\lambda} = 1/2$ only if $n\nu_1 = i\nu_2$.

($\Leftarrow$) Suppose that $n\nu_1 = i\nu_2$, for some $i \in \left\{1, \ldots, \left\lfloor\frac{L-r}{2r}\right\rfloor\right\}$. Then $\frac{L-r}{n\nu_1} = \frac{L-r}{i\nu_2}$ and thus we are in the case for (16b) which evaluates to $\tilde{\lambda} = 1/2$. □

We end this subsection by considering the case when there are insufficient pings, i.e., $n < \frac{L-r}{2r}$. If $\nu_1 \geqslant \frac{2rn}{L-r}\nu_2$, then $\theta(\nu_1, \tau) = \theta(\nu_2, \tau) = \frac{1}{L}(r+2rn) < 1$ on the nonempty interval $[\frac{2r}{\nu_1}, \frac{L-r}{n\nu_2}]$ and the optimum value of $\mathbb{E}_\nu[\theta(\nu, \tau)]$ is achieved for any $\frac{2r}{\nu_1} \leqslant \tau \leqslant \frac{L-r}{n\nu_2}$. Suppose instead that $\nu_1 < \frac{2rn}{L-r}\nu_2$. Similar to the case of sufficient pings, it can be shown that $\frac{L-r}{n\nu_2} \leqslant \tau^* \leqslant \frac{2r}{\nu_1}$. Recall that $\theta(\nu_2, \tau)$ has a maximum value of $\frac{1}{L}(r + 2rn) < 1$. As a result, the decreasing segments of $\theta(\nu_2, \tau)$ do not align with $\left(\frac{L-r}{n\nu_2}, \theta(\nu_2, \tau)\right)$. Hence there does not exist a linear function $\hat{\theta}(\nu_2, \tau)$ as described in Lemma 1, nor does there always exist a threshold value $\tilde{\lambda}$ as described in Theorem 1. Therefore there may exist a unique optimal pining period τ^* within the interval $(\frac{L-r}{n\nu_2}, \frac{2r}{\nu_1})$. However, for certain values of λ, the optimal pinging period, τ^*, can be determined a priori, as shown in Proposition 3.

Proposition 3 *If there are insufficient pings, i.e., $n < \frac{L-r}{2r}$, and the faster speed is more likely than the slower speed, i.e, $\lambda < 1/2$, then $\tau^* = \frac{L-r}{n\nu_2}$.*

Proof. The result comes directly from using the upper bound $\overline{\theta}(\nu_2, \tau)$ as in Corollary 1 and the fact that $\frac{L-r}{n\nu_2}$ is the left endpoint of the interval $\left[\frac{L-r}{n\nu_2}, \frac{2r}{\nu_1}\right]$. □

4.2 Three or More Search Speeds

Suppose there are more than two possible search speeds, i.e., $m > 2$. In the discussion that follows, we consider the case when there are sufficient pings, i.e., $n \geqslant \frac{L-r}{2r}$.

Theorem 2 constructs a continuous piecewise linear upper bound for $\mathbb{E}_\nu[\theta(\nu, \tau)]$ and determines threshold values of the probability weights $g(\nu_1)$ and $g(\nu_m)$ that imply either $\tau^* = \frac{L-r}{n\nu_1}$ or $\tau^* = \frac{2r}{\nu_m}$, respectively.

Theorem 2 *If there are sufficient pings, there exist threshold values $\overline{\lambda}$ and $\underline{\lambda}$ such that when $g(\nu_m) \geqslant \overline{\lambda}$, $\tau^* = \frac{2r}{\nu_m}$, and when $g(\nu_1) \geqslant \underline{\lambda}$, $\tau^* = \frac{L-r}{n\nu_1}$.*

Proof. The functions $\theta(\nu_j, \tau)$, $j = 1, \ldots, m$ are nondecreasing for $\tau < 2r/\nu_m$ and nonincreasing for $\tau > \frac{L-r}{n\nu_1}$. As a convex combination of these functions, $\mathbb{E}_\nu[\theta(\nu, \tau)]$ is nondecreasing and nonincreasing on these intervals, respectively. Thus $\frac{2r}{\nu_m} \leqslant \tau^* \leqslant \frac{L-r}{n\nu_1}$.

For a pinging period $\tau \in \left[\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right]$, define the following sets of search speed indices: $J_a(\tau) := \left\{j : \tau < \frac{L-r}{n\nu_j}\right\}$, $J_b(\tau) := \left\{j : \frac{L-r}{n\nu_j} \leqslant \tau \leqslant \frac{2r}{\nu_j}\right\}$, and $J_c(\tau) := \left\{j : \tau > \frac{2r}{\nu_j}\right\}$, where $j \in \{1, \ldots, m\}$. That is, $J_a(\tau)$, $J_b(\tau)$, and $J_c(\tau)$ are the sets of search speed indices whose probabilities of detection at that pinging period, $\theta(\nu_j, \tau)$, are (a) increasing, (b) constant and maximized, or (c) bounded above by $\hat{\theta}(\nu_j, \tau)$, respectively, at τ. Clearly $J_a(\tau)$, $J_b(\tau)$, and $J_c(\tau)$ are a partition of $\{1, \ldots, m\}$ and it is also easily seen that $J_c\left(\frac{2r}{\nu_m}\right) = \emptyset$ and $J_a\left(\frac{L-r}{n\nu_1}\right) = \emptyset$.

For a pinging period τ, we can bound the expected probability of detection,

$$\begin{aligned} \mathbb{E}_\nu[\theta(\nu, \tau)] &= \sum_{j \in J_a(\tau)} \theta(\nu_j, \tau) g(\nu_j) + \sum_{j \in J_b(\tau)} \theta(\nu_j, \tau) g(\nu_j) + \sum_{j \in J_c(\tau)} \theta(\nu_j, \tau) g(\nu_j) \\ &\leqslant \sum_{j \in J_a(\tau)} \theta(\nu_j, \tau) g(\nu_j) + \sum_{j \in J_b(\tau)} \theta(\nu_j, \tau) g(\nu_j) + \sum_{j \in J_c(\tau)} \hat{\theta}(\nu_j, \tau) g(\nu_j) =: \widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)], \end{aligned} \tag{19}$$

where for each $j \in J_c(\tau)$, we fix a common $\tau' = \frac{L-r}{n\nu_1}$ and thereby bound each $\theta(\nu_j, \tau)$ above by $\hat{\theta}(\nu_j, \tau)$ on the interval $\left[\frac{2r}{\nu_j}, \frac{L-r}{n\nu_1}\right]$. From (19), it is evident that $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ is continuous and piecewise linear, as well as differentiable on $\left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right)$ except at a finite number of pinging periods where a search speed index either transitions from $J_a(\tau)$ to $J_b(\tau)$ or from $J_b(\tau)$ to $J_c(\tau)$.

Observe that for pinging periods $\tau \in \left[\frac{2r}{\nu_j}, \frac{L-r}{n\nu_1}\right]$, the line segment $\hat{\theta}(\nu_j, \tau)$ can be written as

$$\hat{\theta}(\nu_j, \tau) = \theta\left(\nu_j, \frac{2r}{\nu_j}\right) - \frac{\theta\left(\nu_j, \frac{2r}{\nu_j}\right) - \theta\left(\nu_j, \frac{L-r}{n\nu_1}\right)}{\frac{L-r}{n\nu_1} - \frac{2r}{\nu_j}} \left(\tau - \frac{2r}{\nu_j}\right).$$

Because there are sufficient pings, by (4b), $\theta\left(\nu_j, \frac{2r}{\nu_j}\right) = 1$ and so

$$\hat{\theta}(\nu_j, \tau) = 1 - \frac{1 - \theta\left(\nu_j, \frac{L-r}{n\nu_1}\right)}{\frac{L-r}{n\nu_1} - \frac{2r}{\nu_j}} \left(\tau - \frac{2r}{\nu_j}\right). \tag{20}$$

From the definition of $\hat{\theta}(\nu_j, \tau)$, $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)] = \mathbb{E}_\nu[\theta(\nu, \tau)]$ at $\tau = 2r/\nu_m$, where no upper bounds are applied, and at $\tau = \frac{L-r}{n\nu_1}$, where $\left|J_c\left(\frac{L-r}{n\nu_1}\right)\right|$ upper bounds are applied. Substituting expressions for $\theta(\nu_j, \tau)$ and $\hat{\theta}(\nu_j, \tau)$ from (4a), (4b) and (20) into (19), we have

$$\begin{aligned} \widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)] = &\sum_{j \in J_a(\tau)} \left(\frac{r + n\nu_j\tau}{L}\right) g(\nu_j) + \sum_{j \in J_b(\tau)} (1) g(\nu_j) \\ &+ \sum_{j \in J_c(\tau)} \left(1 - \frac{1 - \theta\left(\nu_j, \frac{L-r}{n\nu_1}\right)}{\frac{L-r}{n\nu_1} - \frac{2r}{\nu_j}} \left(\tau - \frac{2r}{\nu_j}\right)\right) g(\nu_j). \end{aligned} \tag{21}$$

For $j \in J_c(\tau)$, define

$$p_j := \frac{1 - \theta\left(\nu_j,\, \frac{L-r}{n\nu_1}\right)}{\frac{L-r}{n\nu_1} - \frac{2r}{\nu_j}} > 0,$$

the *negative* slope of the upper bound $\hat{\theta}(\nu_j, \tau)$, as shown in (20). When a pinging period $\tau \in \left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right)$ is not a breakpoint of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$, differentiating (21) with respect to τ gives the slope of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ at τ,

$$\frac{\partial}{\partial \tau} \widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)] = \sum_{j \in J_a(\tau)} \frac{n\nu_j}{L} g(\nu_j) - \sum_{j \in J_c(\tau)} p_j g(\nu_j). \tag{22}$$

As the pinging period τ increases, the cardinality of $J_a(\tau)$ is nonincreasing while that of $J_c(\tau)$ is nondecreasing. Thus it follows from (22) that the slopes of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ are nonincreasing as τ increases.

Define

$$\overline{\tau} := \sup \left\{ \tau \in \left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right) : J_a(\tau) = J_a\left(\frac{2r}{\nu_m}\right),\, J_b(\tau) = J_b\left(\frac{2r}{\nu_m}\right) \backslash \{m\},\, J_c(\tau) = \{m\} \right\},$$

the leftmost breakpoint of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ on $\left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right)$. If the slope of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ on the subinterval $\left(\frac{2r}{\nu_m}, \overline{\tau}\right)$ is nonpositive, then all of the other slopes are nonpositive and $\tau^* = \frac{2r}{\nu_m}$. Fix $0 < \epsilon < \overline{\tau} - \frac{2r}{\nu_m}$ so that $\frac{2r}{\nu_m} + \epsilon$ lies in $\left(\frac{2r}{\nu_m}, \overline{\tau}\right)$ and thus $J_a\left(\frac{2r}{\nu_m} + \epsilon\right) = J_a\left(\frac{2r}{\nu_m}\right)$ and $J_c\left(\frac{2r}{\nu_m} + \epsilon\right) = \{m\}$. By substitution into (22),

$$\begin{aligned}
\sum_{j \in J_a\left(\frac{2r}{\nu_m} + \epsilon\right)} \frac{n\nu_j}{L} g(\nu_j) - \sum_{j \in J_c\left(\frac{2r}{\nu_m} + \epsilon\right)} p_j g(\nu_j) \leqslant 0 \quad &\iff \quad \sum_{j \in J_a\left(\frac{2r}{\nu_m}\right)} \frac{n\nu_j}{L} g(\nu_j) - p_m g(\nu_m) \leqslant 0 \\
&\iff \quad g(\nu_m) \geqslant \frac{n}{Lp_m} \sum_{j \in J_a\left(\frac{2r}{\nu_m}\right)} \nu_j g(\nu_j) =: \overline{\lambda}.
\end{aligned}$$

On the other hand, define

$$\underline{\tau} := \inf \left\{ \tau \in \left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right) : J_a(\tau) = \{1\},\, J_b(\tau) = J_b\left(\frac{L-r}{n\nu_1}\right) \backslash \{1\},\, J_c(\tau) = J_c\left(\frac{L-r}{n\nu_1}\right) \right\},$$

the rightmost breakpoint of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ on $\left(\frac{2r}{\nu_m}, \frac{L-r}{n\nu_1}\right)$. If the slope of $\widehat{\mathbb{E}}_\nu[\theta(\nu, \tau)]$ on the subinterval $\left(\underline{\tau}, \frac{L-r}{n\nu_1}\right)$ is nonnegative, then all of the other slopes are nonnegative and $\tau^* = \frac{L-r}{n\nu_1}$. Again, fix

$0 < \epsilon < \frac{L-r}{n\nu_1} - \underline{\tau}$ so that $\frac{L-r}{n\nu_1} - \epsilon$ lies in $\left(\underline{\tau},\, \frac{L-r}{n\nu_1}\right)$ and thus $J_a\left(\frac{L-r}{n\nu_1} - \epsilon\right) = \{1\}$ and $J_c\left(\frac{L-r}{n\nu_1} - \epsilon\right) = J_c\left(\frac{L-r}{n\nu_1}\right)$. By substitution into (22),

$$\sum_{j \in J_a\left(\frac{L-r}{n\nu_1} - \epsilon\right)} \frac{n\nu_j}{L} g(\nu_j) - \sum_{j \in J_c\left(\frac{L-r}{n\nu_1} - \epsilon\right)} p_j g(\nu_j) \geqslant 0 \iff \frac{n\nu_1}{L} g(\nu_1) - \sum_{j \in J_c\left(\frac{L-r}{n\nu_1}\right)} p_j g(\nu_j) \geqslant 0$$

$$\iff g(\nu_1) \geqslant \frac{L}{n\nu_1} \sum_{j \in J_c\left(\frac{L-r}{n\nu_1}\right)} p_j g(\nu_j) =: \underline{\lambda}. \quad \square$$

Corollary 4 now proves that there is not a unique optimal pinging period between the endpoints of the leftmost subinterval $\left(\frac{2r}{\nu_m},\, \overline{\tau}\right)$.

Corollary 4 *The interval* $\left(\frac{2r}{\nu_m},\, \overline{\tau}\right)$ *does not contain a unique* τ^*.

Proof. Suppose that $\tau^* \in \left(\frac{2r}{\nu_m},\, \overline{\tau}\right)$. We apply a linear upper bound $\hat{\theta}(\nu_m,\, \tau)$ with a fixed right endpoint of $\tau' = \overline{\tau}$, so that for pinging periods in $\left[\frac{2r}{\nu_m},\, \overline{\tau}\right]$,

$$\mathbb{E}_\nu[\theta(\nu,\, \tau)] \leqslant \sum_{j \in J_a(\overline{\tau})} \theta(\nu_j,\, \tau) g(\nu_j) + \sum_{j \in J_b(\overline{\tau})} \theta(\nu_j,\, \tau) g(\nu_j) + \hat{\theta}(\nu_m,\, \tau) g(\nu_m) =: \widehat{\mathbb{E}}_\nu[\theta(\nu,\, \tau)],$$

with equality at $\tau = \frac{2r}{\nu_m}$ and $\tau = \overline{\tau}$. Because $\widehat{\mathbb{E}}_\nu[\theta(\nu,\, \tau)]$ is linear on the interval $\left[\frac{2r}{\nu_m},\, \overline{\tau}\right]$, it attains its maximum at one of the endpoints. Without loss of generality, suppose this occurs at the left endpoint $\tau = \frac{2r}{\nu_m}$. Then

$$\mathbb{E}_\nu[\theta(\nu,\, \tau^*)] \leqslant \widehat{\mathbb{E}}_\nu[\theta(\nu,\, \tau^*)] \leqslant \widehat{\mathbb{E}}_\nu\left[\theta\left(\nu,\, \frac{2r}{\nu_m}\right)\right] = \mathbb{E}_\nu\left[\theta\left(\nu,\, \frac{2r}{\nu_m}\right)\right],$$

which is a contradiction because τ^* is the unique optimal value of τ. A similar contradiction results if $\widehat{\mathbb{E}}_\nu\left[\theta\left(\nu,\, \tau\right)\right]$ attains its maximum at the right endpoint $\tau = \overline{\tau}$. $\square$

5 Case Study: Current ULB Technology and Search Strategy

We consider current state-of-the-art technology for ULBs and apply the main results of Section 3 to real-world searches for FDRs. A recent incident that is representative of modern search strategy for missing aircraft is the search for Malaysia Airlines Flight 370 (MH370) which disappeared on March 8, 2014 when it lost contact with ground control en route from Kuala Lumpur to Beijing (Malaysia

Airlines, 2014). The Malaysian government began search and rescue operations in the South China Sea and Gulf of Thailand and later in the Strait of Malacca, before satellite and military radar data suggested that the plane most likely turned and traveled south over the Indian Ocean (Hildebrandt, 2014; ABC News, 2014). When preliminary air and sea searches in the south Indian Ocean found no debris matching the airplane, *Ocean Shield* of Australia and *HMS Echo* of the United Kingdom were deployed to locate the FDR within a revised search area of 221,000 square kilometers, located 1,500 kilometers off the west coast of Australia (Joint Agency Coordination Center, 2014a,b). Within this search area, the ocean depth is between 10,000 and 15,000 feet, well below the maximum operating depth of 20,000 feet for ULBs (Associated Press, 2014; Radiant Power Corp, 2014; RJE International, 2014). Depending on factors such as maintenance and temperature, the ULB for MH370 likely went silent around April 12, 2014, a little over a month after the aircraft went missing (Associated Press, 2014).

When towing pinger locators, search vessels such as *Ocean Shield* and *HMS Echo* travel in parallel linear paths, as described in Section 1, at speeds between 1.9 and 9.3 kilometers per hour, depending on the ocean depth (United States Navy, 2013). The grid-like search path of the search vessels can be viewed as a linear search trajectory over a lengthy interval. An industry standard ULB, such as that found on MH370, can be detected within a radius of 1–2 kilometers in normal conditions and 4–5 kilometers in good conditions (Kelland, 2009).

Taking the fastest speed ($\nu = 9.3$ km/h) and shortest range ($r = 1$ km) and assuming a uniform beacon location distribution, Proposition 1 implies a conservative estimate of no less than 12.9 minutes for the optimal pinging period. This estimate is approximately 700 times longer than the industry-standard pinging period of 1.11 seconds, based on a ping repetition rate of 0.9 pings per second (Teledyne Benthos, 2011). For the industry-standard pinging period to be optimal, Proposition 1 suggests that the speed of the search vessel would have to be around 6,500 kilometers per hour, over 12 times the speed of the fastest known boat, *Spirit of Australia*, (Australian National Maritime Museum, 1991). While our assumptions—a definite range law for TPL detection and the search terminating after one detected ping—simplify the realities of deep-sea searches with TPLs, we believe they alone cannot explain away the magnitude of the difference between the industry-standard pinging period and our conservative estimate of the optimal pinging period. Therefore, for the purposes of open-ocean searches for flight data recorders, our analysis suggests that the current pinging period is much too short.

6 Conclusions

Motivated by underwater searches for lost flight data recorders, we model the search for an immobile beacon that is only detectable during a finite number of pings. We consider the optimal pinging period under discrete distributions over the search speed. We derive closed-form solutions for the optimal pinging period for special cases, and provide bounds on the optimal pinging periods for the more general problem. Given a fixed search speed, we determine the optimal pinging frequency, and conclude that the currently used pinging period of one second is far too short.

Future research directions include more general distributions of the beacon location as well as continuous distributions over search speeds. Relaxing the assumption of a definite range law to consider probabilistic detection functions such as an inverse cube law or exponential attenuation would also enhance the model. As turning a search vessel can require many hours, another promising research area is to model the possibility of changing directions.

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